

INTRODUCTION TO QUANTUM ELECTRODYNAMICS

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I. The interaction of electromagnetic fields with matter.

The Lagrangian for the charge q in electromagnetic potentials V and \vec{A} is,

$$L = \frac{1}{2}m\vec{v}^2 - qV + \frac{q}{c}\vec{A} \cdot \vec{v},$$

where the interaction is through the scalar potential (qV) and the vector potential ($\vec{J} \cdot \vec{A} = q\vec{v} \cdot \vec{A}$); the fields in terms of the potentials are,

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \vec{E} = -\vec{\nabla}V - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}. \quad (1)$$

Canonical momenta are defined by,

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{q}{c}A_i,$$

or

$$\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}.$$

The Hamiltonian can thus be written as $H = \sum_i p_i \dot{x}_i - L$, or

$$H = \frac{1}{2m}(\vec{p} - q\vec{A}/c)^2 + qV. \quad (2)$$

To go over to quantum mechanics, one turns the canonical momentum into a quantum operator by the transcription $\vec{p} \rightarrow \hbar/i\vec{\nabla}$.

II. Maxwell's equations in free space and their solutions.

The Maxwell equations in free space (Gaussian units) are,

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 & \vec{\nabla} \cdot \vec{B} &= 0, \\ \vec{\nabla} \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} & \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}. \end{aligned}$$

The two equations on the left imply that in free space the (time-dependent) fields can be written in terms of the potentials, as in (1), but where the scalar potential is zero and

(Coulomb Gauge) $\vec{\nabla} \cdot \vec{A} = 0$. The Hamiltonian for the *fields only* can therefore be written in terms of the vector potential alone as,

$$\begin{aligned} H_{\text{fields}} &= \frac{1}{8\pi} \int d^3r (\vec{E}^2 + \vec{B}^2) \\ &= \frac{1}{8\pi} \int d^3r \left[\left(-\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right)^2 + (\vec{\nabla} \times \vec{A})^2 \right]. \end{aligned} \quad (3)$$

The second pair of the Maxwell equations imply that the vector potential satisfies the wave equation,

$$\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} = 0. \quad (4)$$

Solutions of (4) are plane waves with $\omega = ck$:

$$\vec{A}(\vec{r}, t) = \vec{A}_0 e^{\pm i(\vec{k} \cdot \vec{r} - \omega t)}.$$

The general solution in free space is therefore,

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \left[C_{k, \lambda} \vec{\epsilon}_{k, \lambda} e^{i(\vec{k} \cdot \vec{r} - \omega t)} + C_{k, \lambda}^* \vec{\epsilon}_{k, \lambda} e^{-i(\vec{k} \cdot \vec{r} - \omega t)} \right] / \sqrt{V},$$

where the wavevector and polarization vectors form an orthogonal triplet,

$$\vec{k} \cdot \vec{\epsilon}_{k, \lambda} = 0, \quad \vec{\epsilon}_{k, 1} \cdot \vec{\epsilon}_{k, 2} = 0, \quad \vec{\epsilon}_{k, 1} \times \vec{\epsilon}_{k, 2} = \vec{k}/k.$$

For convenience, define new dimensionless coefficients a, a^* from,

$$C_{k, \lambda} = c \sqrt{\frac{2\pi\hbar}{\omega}} a_{k, \lambda}.$$

Then the general solution of the wave equation becomes,

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k}, \lambda} c \sqrt{\frac{2\pi\hbar}{\omega V}} \left[a_{k, \lambda} \vec{\epsilon}_{k, \lambda} e^{i(\vec{k} \cdot \vec{r} - \omega t)} + \text{c.c.} \right], \quad (5)$$

together with the associated orthonormality of the plane waves,

$$\int d^3r e^{i\vec{k} \cdot \vec{r}} e^{-i\vec{k}' \cdot \vec{r}} = V \delta_{\vec{k}, \vec{k}'}$$

Substitution of (5) into (3) results in a very simplified form of the Hamiltonian,

$$H_{\text{fields}} = \sum_{k, \lambda} \left(a_{k, \lambda}^* a_{k, \lambda} + \frac{1}{2} \right) \hbar \omega. \quad (6)$$

III. Photons

To quantize the theory defined by (5) and (6) we promote the a -coefficients to creation and destruction operators,

$$a_{k,\lambda} \rightarrow \hat{a}_{k,\lambda} \quad a_{k,\lambda}^* \rightarrow \hat{a}_{k,\lambda}^\dagger,$$

equipped with Bose commutation relations,

$$[\hat{a}_{k,\lambda}, \hat{a}_{k',\lambda'}^\dagger] = \delta_{k,k'} \delta_{\lambda,\lambda'}.$$

The Hamiltonian then becomes a sum of number operators counting photon numbers of differing wavevector and polarization. The infinite "zero-point" energy which is the sum over $1/2$ is discarded as unphysical. States of an indefinite number of photons ("Fock states") are defined from the action of the creation and destruction operators:

$$\text{general state: } |n_{\vec{k}_1, \lambda_1}, n_{\vec{k}_2, \lambda_2}, \dots\rangle,$$

$$\hat{a}_{k,\lambda} |n_{k,\lambda}\rangle = \sqrt{n_{k,\lambda}} |n_{k,\lambda} - 1\rangle$$

$$\hat{a}_{k,\lambda}^\dagger |n_{k,\lambda}\rangle = \sqrt{n_{k,\lambda} + 1} |n_{k,\lambda} + 1\rangle.$$

Other physical observables may also of course be written in terms of the new operators; for example, the electromagnetic momentum operator,

$$\vec{P} = \frac{1}{4\pi c} \int d^3r (\vec{E} \times \vec{B}) = \sum_{k,\lambda} \hbar \vec{k} \hat{a}_{k,\lambda}^\dagger \hat{a}_{k,\lambda}.$$

The complete Hamiltonian, which treats the atom only nonrelativistically, is then,

$$\begin{aligned} \hat{H} &= \frac{1}{2m_e} (\vec{p} - \frac{e}{c} \vec{A})^2 - \frac{e^2}{r^2} + \frac{1}{8\pi} \int d^3r \left[\left(\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right)^2 + (\vec{\nabla} \times \vec{A})^2 \right] \\ &\equiv \hat{H}_0 + \hat{H}_{\text{int}}, \end{aligned} \quad (7)$$

where

$$\hat{H}_0 = \left(-\frac{\hbar^2}{2m_e} \vec{\nabla}^2 - \frac{e^2}{r^2} \right) + \hat{H}_{\text{field}},$$

whose eigenstates (without electron spin) are,

$$|n, \ell, m\rangle |\text{photon numbers}\rangle,$$

and the interaction Hamiltonian is,

$$\hat{H}_{\text{int}} = \frac{e}{m_e c} \vec{A} \cdot \frac{\hbar}{i} \vec{\nabla} + \frac{e^2}{2m_e c^2} \vec{A}^2. \quad (8)$$

Equation (8) follows because of the identity $\vec{\nabla} \cdot (\vec{A}\psi) = (\vec{\nabla} \cdot \vec{A})\psi + (\vec{A} \cdot \vec{\nabla})\psi$ and that we are working in a gauge where $\vec{\nabla} \cdot \vec{A} = 0$. Also, the above neglects the interaction of the electron and proton fields which is down by a factor of m_e/m_p .

IV. Perturbation Theory

The interactions given in (8) are sufficient to calculate quantum electrodynamic effects. Here, we will be content with first-order, time-dependent theory which results in the Fermi Golden Rule approximation for the transition rate between two quantum states labelled i and f ,

$$\Gamma = \sum_{\lambda} \int \frac{2\pi}{\hbar} |\langle f | \hat{H}_i | i \rangle|^2 \rho d\epsilon \delta(\epsilon_f - \epsilon_i \pm \hbar\omega). \quad (9)$$

The density of states ρ for photons required is,

$$\rho(\epsilon)d\epsilon = \frac{V}{(2\pi)^3} \frac{\omega^2}{c^3} d\omega d\Omega. \quad (10)$$

V. Spontaneous Emission

The states of the unperturbed hydrogen atom $|n, \ell, m\rangle$ are stationary; if left alone the states will remain forever. However, in the *quantum* theory of fields governed by (6) and (8) there are always photons present even in a vacuum; their presence may cause spontaneous emission from an excited initial state. In particular, the first order term \vec{A} in \hat{H}_i contains a linear combination of creation and destruction operators and thus creates or destroys a *single* photon. The \vec{A}^2 term contains quadratic combinations of the photon operators and can thus contribute to *two* photon processes. Note that the exact time-development operator in the interaction picture is the time-ordered exponential,

$$\hat{U}(t) = T \exp\left\{-\frac{i}{\hbar} \int_0^t \hat{H}_i(t') dt'\right\},$$

which has all possible numbers of photons created or destroyed.

Let us now consider a spontaneous transition between a $2P$ state and the $1S$ ground state of the hydrogen atom. Then $|i\rangle = |2, 1, m\rangle$, and $|f\rangle = |1, 0, 0\rangle$. The essential matrix element needed for first order theory is,

$$\begin{aligned} \langle f | \hat{H}_i | i \rangle &= \langle 1_{k,\lambda} | \langle 1, 0, 0 | \hat{H}_i | 2, 1, m \rangle | 0 \rangle \\ &= \frac{e}{m_e} \sqrt{\frac{2\pi\hbar}{\omega V}} \int d^3r \Psi_{100}^*(r) e^{-i\vec{k}\cdot\vec{r}} \vec{\epsilon}_{k,\lambda} \cdot \frac{\hbar}{i} \vec{\nabla} \Psi_{210}(r) \langle 1_{k,\lambda} | \hat{a}_{k,\lambda}^\dagger | 0 \rangle. \end{aligned} \quad (11)$$

Note that if there were one photon initially present, the photon part for *stimulated* emission would be $\langle 2_{k,\lambda} | \hat{a}_{k,\lambda}^\dagger | 1_{k,\lambda} \rangle = \sqrt{2}$, and thus the stimulated emission rate (to first order) would be larger by a factor of 2 from the spontaneous emission rate that we obtain below.

VI. Spontaneous emission rate in the electric dipole approximation

For *optical* transitions in the range of $4000 - 6000 \text{ \AA}$, such as the $2P \rightarrow 1S$ transition in hydrogen, $\vec{k} \cdot \vec{r} \ll 1$, so that one may make the approximation $\exp\{-i\vec{k} \cdot \vec{r}\} \simeq 1$ ($r \simeq a_0 \simeq 1 \text{ \AA}$). Thus, we may write the transition rate per unit solid angle, for polarization λ , as,

$$\frac{d^2\Gamma_\lambda}{d\Omega d\omega} = \frac{2\pi}{\hbar} \left(c \sqrt{\frac{2\pi\hbar}{\omega V}} \right)^2 \left| \int d^3r \Psi_{100}^* \vec{\epsilon}_{k,\lambda} \cdot \vec{p} \Psi_{21m} \right|^2 \left(\frac{e}{mc} \right)^2 \rho.$$

The matrix element of the momentum operator may be conveniently computed as follows:

$$\vec{p} = \frac{im}{\hbar} [\vec{p}^2/2m - e^2/r, \vec{r}], \quad \text{and thus}$$

$$\begin{aligned} \langle 100 | \vec{p} | 21m \rangle &= \frac{im}{\hbar} \langle [H_0, \vec{r}] \rangle \\ &= \frac{im}{\hbar} (E_2 - E_1) \langle \vec{r} \rangle \\ &= i\omega m \langle \vec{r} \rangle \end{aligned}$$

We now need to compute,

$$\begin{aligned} \frac{d\Gamma_\lambda}{d\Omega} &= \frac{2\pi}{\hbar} \int d\omega \left(c \sqrt{\frac{2\pi\hbar}{\omega V}} \right)^2 \frac{e^2 \omega^2}{c^2} \left| \int d^3r \Psi_{100}^* \vec{\epsilon}_{k,\lambda} \cdot \vec{r} \Psi_{21m} \right|^2 \times \\ &\quad \times \left(\frac{V}{2\pi} \right)^2 \frac{\omega^2}{\hbar c^3} \delta(\epsilon_f - \epsilon_i - \hbar\omega). \end{aligned} \quad (12)$$

For unpolarized photons, and random values of $m_\ell = 0, \pm 1$, we average over all possible initial states: we take $\frac{1}{3} \sum_{m=0,\pm 1}$. Also,

$$\begin{aligned} \vec{r} \cdot \vec{\epsilon} &= \sum_{q=0,\pm 1} r_q \epsilon_{-q} (-1)^q \\ &= r \sqrt{\frac{4\pi}{3}} \left(\frac{\epsilon_x + i\epsilon_y}{\sqrt{2}} Y_{1,-1} - \frac{\epsilon_x - i\epsilon_y}{\sqrt{2}} Y_{1,1} + \epsilon_z Y_{1,0} \right), \end{aligned}$$

and

$$Y_{1,1} = -Y_{1,-1} \quad Y_{1,0} = \frac{1}{\sqrt{4\pi}} \quad \int d\Omega Y_{1,m}^* Y_{1,m'} = \delta_{mm'}.$$

The integral above is therefore,

$$\int d^3r R_{1,0} \frac{r}{\sqrt{4\pi}} \sqrt{\frac{4\pi}{3}} \text{ (same as above) } R_{2,1} Y_{1,m}$$

$$= \sqrt{1/3} \left(-\frac{\epsilon_x + i\epsilon_y}{\sqrt{2}} \delta_{m,1} + \frac{\epsilon_x - i\epsilon_y}{\sqrt{2}} \delta_{m,-1} + \epsilon_z \delta_{m,0} \right) \int_0^\infty dr r^3 R_{1,0} R_{2,1}.$$

The last radial integral has the value $\sqrt{3/2}(2^8/3^5)a_0$. So taking $\frac{1}{3} \sum_m | \cdot |^2$ in the last expression yields

$$\frac{2^{15}}{3^{10}} a_0^2 \cdot \frac{1}{3} (\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) = \frac{2^{15}}{3^{11}} a_0^2.$$

Putting together all of the above pieces into (12) and restoring the sum over polarizations produces,

$$\Gamma_{2p \rightarrow 1s} = \sum_\lambda \int d\Omega \frac{\alpha \omega^3}{2\pi c^2} \frac{2^{15}}{3^{11}} a_0^2$$

$$= \frac{\alpha \omega^3}{c^2} \frac{2^{17}}{3^{11}} a_0^2,$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant. The final result for the transition probability is,

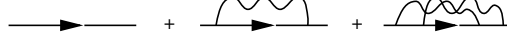
$$\Gamma_{2p \rightarrow 1s} = \left(\frac{2}{3}\right)^8 \alpha^5 \left(\frac{mc^2}{\hbar}\right) \simeq 0.6 \times 10^9 \text{ Hz.}$$

The mean transition time for spontaneous emission from $2P$ to $1S$ is,

$$\boxed{\tau_{2p \rightarrow 1s} = \Gamma_{2p \rightarrow 1s}^{-1} = 1.6 \times 10^{-9} \text{ sec.}}$$

VII. Q.E.D. processes: Feynman Diagrams

Electron Self Energy:



$e^+ e^-$ Scattering:

