What is a Tensor?

A **tensor** is a set of numbers which transforms under rotations in according to the rule,

$$T'_i = R_{i,j} T_j, \tag{1}$$

where i=1,2, (1,2,3 in three dimensions) and the repeated index j implies summation over $j = 1, 2$. The matrix R is the rotation matrix which rotates the co-ordinate axes through a given angle θ in the positive sense,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

Thus, a *vector* is a tensor of rank one (one set of indices). A tensor of rank 2 has two sets of indices, so requires two R-matrices one for each index,

$$T'_{i,j} = R_{i,k} R_{j,\ell} T_{k,\ell}.$$

Here, summation over k, ℓ is again implied. The latter equation can also be written in matrix form as,

$$T' = RT\tilde{R}.$$

How do you know whether or not a set of "objects" is a tensor? Is the set of functions $W_{1,1} = x^2,\ W_{1,2} = W_{2,1} = xy,\ W_{2,2} = y^2$ a tensor of rank two? Let c denote $\cos\theta$ and s denote $\sin\theta$. Then according to the above transformation law, one ought to have

$$W' = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix},$$

or,

$$W' = \begin{pmatrix} c^2x^2 + 2scxy + s^2y^2 & -scx^2 + scy^2 + (c^2 - s^2)xy \\ -scx^2 + scy^2 + (c^2 - s^2)xy & s^2x^2 - 2scxy + c^2y^2 \end{pmatrix}. \tag{3}$$

If W' is a tensor, (3) shows how - by definition - it must transform under co-ordinate rotations. Does it? According to the geometrical formulas (2), we must have

$$x'^2 = (cx + sy)^2 = c^2x^2 + 2csxy + s^2y^2,$$

$$x'y' = -scx^2 + scy^2 + (c^2 - s^2)xy,$$

$$y'^2 = s^2x^2 - 2scxy + c^2y^2.$$

Thus, using these results, (3) finally reads,

$$W' = \begin{pmatrix} x'^2 & x'y' \\ x'y' & y'^2 \end{pmatrix}.$$

From which it is easily seen that in the new co-ordinate system W takes exactly the same form as it did in the old system. That is the advantage of casting physical law into tensor form; the form of the equations remains co-ordinate invariant.