INTRODUCTION TO CONSTRAINED VARIATION

Often one is faced with problems in which some quantity must be extremized (maximized or minimized) while a second quantity is held at a fixed value. Such constrained variational calculations are handled by the method of Lagrange multipliers. Suppose for example one needs to find the extremum of,

\[ A[F] = \int_a^b F(\dot{x}, x, t) \, dt \]

over all functions \( x(t) \) which satisfy \( G[x(t)] = 0 \). The functional \( A[F] \) is then modified to include the constraint with the addition of the multiplier, \( \lambda \) as,

\[ \tilde{A}[F] = \int_a^b F(\dot{x}, x, t) \, dt + \lambda(G[x(t)]). \] (1)

Differentiation with respect to \( \lambda \) will reproduce the constraint, while the space of functions \( x(t) \) is probed by functional variation of \( \tilde{A} \) with respect to \( x \),

\[ 0 = \frac{\delta \tilde{A}}{\delta x} = \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} - \frac{\partial F}{\partial x} + \lambda \frac{\delta G}{\delta x}, \] (2)

\[ 0 = \frac{\partial \tilde{A}}{\partial \lambda} = G[x]. \] (3)

Physicists will recognized (1)-(3) as Lagrangian mechanics with \( F = T - V \).

**EXAMPLE 1** In many applications of physics, mathematical physics and engineering the principle of Maximum Entropy is very useful. Suppose one wishes to find a probability distribution \( P(x) \) with a given mean \( x_1 \) and average square \( x_2 \). The constrained functional (1) is given by,

\[ \tilde{A} = \int_0^1 P(x) \log P(x) \, dx + \lambda_1(x_1 - \int_0^1 dx \, xP(x)) + \lambda_2(x_2 - \int_0^1 dx \, x^2 P(x)). \]

Differentiation with respect to the \( \lambda \)'s and variation with respect to \( P \) leads to,

\[ P(x) = \frac{1}{Z} \exp\{-\lambda_1 x + \lambda_2 x^2\}, \]
where \( Z \) is determined by the normalization of \( P \) to unity, and the multipliers are determined by the non-linear constraint equations,

\[
x_1 = \int_0^1 dx xP(x)
\]

\[
x_2 = \int_0^1 dx x^2 P(x),
\]

with \( P(x) \) in the above form.

**EXAMPLE 2** This example is taken from thermodynamics. Consider three identical objects (of any composition) at initial (absolute) temperatures 300, 300, 100. These objects are isolated from all heat sources or sinks. Using only reversible processes, what is the maximum possible temperature to which any one of these objects may be raised?

Let \( T_1, T_2, T_3 \) be the final temperatures of the three objects. Then by the first and second laws of thermodynamics,

\[
\Delta S = \sum_{i=1}^{3} \int \frac{dQ_i}{T_i} = C(\ln \frac{T_1}{300} + \ln \frac{T_2}{300} + \ln \frac{T_3}{100}),
\]

where \( C \) is the specific heat, is to be maximized, subject to the constraint of conservation of energy,

\[
0 = \Delta E = mC[(T_1 - 300) + (T_2 - 300) + (T_3 - 100)].
\]

Maximizing the entropy change \( \Delta S \) while a multiplier enforces the energy constraint yields,

\[
T_1 = 400 \quad T_2 = T_3 = 150,
\]

with of course the symmetric solution with \( T_1 \) and \( T_2 \) interchanged. The details require only ordinary calculus.

**EXAMPLE 3** A cylinder of mass \( M \), radius \( R \) and length \( \ell \) is filled with a *viscous* fluid of fixed density \( \rho \). The cylinder itself is set into rotation of initial frequency \( \Omega_0 \) by a quick twist about the axis of the cylinder (z-axis). Show that at equilibrium the fluid inside rotates in bulk at the final frequency \( \Omega_f \) of the cylinder.

At equilibrium, the total energy of the cylinder-fluid system is,

\[
E = \frac{1}{2} I_{cyl} \Omega_f^2 + \frac{1}{2} \int_{\text{fluid}} \omega^2 dI_{\text{fluid}}
\]

\[
= \frac{1}{2} I_c \Omega_f^2 + \pi \ell \rho \int_0^R r^3 \omega^2(r) dr,
\]
where the cylindrical coordinate from the axis is $r$. The system loses mechanical energy as the fluid elements interact via fluid friction with themselves and with the walls of the container; the energy is therefore minimum at equilibrium. However, angular momentum is conserved in this torque-free system; angular momentum conservation reads,

$$\tilde{L} = I_c \tilde{\Omega}_0 = I_c \Omega_f \tilde{z} + \int_{\text{fluid}} \tilde{r} \times d\tilde{p},$$

or,

$$L_z = \text{constant} = I_c \Omega_f + 2\pi \int_0^R \ell r^3 \omega(r) dr.$$

Minimization of the constrained energy $\tilde{E} = E + \lambda (L_z - I_c \Omega_0)$ by the procedure of (1)-(3) yields $\lambda = \Omega_f$ and $\omega(r) = \Omega_f$ independent of $r$.

**EXAMPLE 4** Consider an *incompressible* mass of uniform density. What is the shape of the surface defining this mass which will produce the largest gravitational field strength $\tilde{g}$ at a point outside of the mass? We will consider the origin to be the point at which we calculate the field strength. It is not too difficult to see that the requisite surface must be a surface of revolution about the z-axis, as in the figure below, which shows the geometry of the surface.

![Diagram of surface of revolution](image)

*The mass surface*
The field strength is,
\[
\vec{g} = \int_V d^3r \frac{\vec{r}}{r^3},
\]
subject to the incompressibility constraint,
\[
\int d^3r = V.
\]
In the above, we have taken the gravitational constant \( G \) to be unity. It is evident from the geometry that any \( \hat{\rho} \) component of the field must cancel yielding only a \( z \)-component whose value is,
\[
g_z = \int_0^{z_0} \frac{\pi \rho(z)^2 z}{(\rho(z)^2 + z^2)^{3/2}} dz.
\]
We must then maximize the constrained field strength
\[
\tilde{g} = \int_0^{z_0} \frac{\pi z \rho(z)^2}{(\rho(z)^2 + z^2)^{3/2}} + \lambda [V - \int_0^{z_0} \pi \rho(z)^2 dz].
\]
Again, the constraint is reproduced by differentiation with \( \lambda \),
\[
0 = \frac{\partial \tilde{g}}{\partial \lambda} = V - \int_0^{z_0} \pi \rho(z)^2 dz.
\]
The constant \( z_0 \) is determined by this condition given \( \rho(z) \) and the fixed volume \( V \). Functional minimization with respect to \( \rho(z) \) gives,
\[
0 = \frac{\delta \tilde{g}}{\delta \rho}, \quad \text{or}
0 = 2z(\rho^2 + z^2) - 3z\rho^2 - 2\lambda(\rho^2 + z^2)^{5/2}.
\]
Progress in solving these can be made by going to spherical coordinates,
\[
\rho = r \sin \theta, \quad z = r \cos \theta,
\]
when the above reduces to,
\[
r^2 = \frac{(2 - 3 \sin^2 \theta) \cos \theta}{2\lambda},
\]
where
\[
\theta \leq \arcsin \sqrt{2/3} \simeq 0.96 \text{ rad}.
\]
It is evident from the above that \( z_0 = 1/\sqrt{\lambda} \). Solving the above for \( r \) and putting the resulting expressions into \( \rho = r \sin \theta \) and \( z = r \cos \theta \) give a set of parametric equations (in \( \theta \)) determining \( \rho = \rho(z) \) for the surface bounding \( V \). The field \( g_z \) at the origin caused by this mass is as large as possible. The curve in the figure is shaped roughly correctly with the slope of 0.96 rad (55 degrees) at the origin and vertical at \( z = z_0 \).